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INFINITE SPECTRA IN THE FIRST ORDER THEORY OF GRAPHS

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The psectrum Spec(A) of a sentence A is, roughly, the set of those a for which A has a threshold function at or near $p=n^{-a}$. Examples are given of A with infinite spectra and with spectra of order type ω^i for arbitrary i.

In their fundamental work On the Evolution of Random Graphs Paul Erdős and Alfréd Rényi [1] showed that many natural graphtheoretic properties A possessed a threshold function $p(n)$, that is, a function having the property the if $r(n)\ll p(n)$ the random graph $G(n, r(n))$ a.s. did not satisfy A while if $p(n)\ll r(n)$ it a.s. did satisfy A. With this as motivation with S. Shelah [2] we defined the spectrum Spec (A) to be, roughly, those α for which there is a threshold function near $n^{-\alpha}$. Precisely, $\alpha\notin$ Spec(A) if there is a positive ε and δ either zero or one so that for any $p(n)$ satisfying $n^{-\alpha-\varepsilon}<p(n)<n^{-\alpha-\varepsilon}$ the probability that A holds in $G(n, p(n))$ tends to δ. For our purposes we note that if for all sufficiently small ε A holds a.s. in $G(n, n, ^{-\alpha+\varepsilon})$ and $\neg A$ a.s. in $G(n, n^{-\alpha-\varepsilon})$, or the reverse, then $\alpha\in$Spec (A).

We restrict our attention to sentences A of the First Order Theory of Graphs. This language contains all Boolean connectives ($\wedge$, $\vee$, $\neg$, ...), an infinite sequence of variables $x, y, z, \ldots$, existential and universal quantification $(\exists x)$, (x) and the predicates equality $(x=y)$ and adjacency $(x\sim y)$. As examples, we may express:
There are no isolated points:

$$(x)(\exists y)(x\sim y).$$

There is a triangle:

$$(\exists x)(\exists y)(\exists z)[x\sim y\wedge x\sim z\wedge y\sim z].$$

Every point lies in an edge not in a triangle:

$$(x)(\exists y)[x\sim y]\wedge(\exists z)[z\sim x\wedge z\sim y].$$

However, many basic graphtheoretic properties such as connectivity, planarity and Hamiltonicity cannot be expressed in this language.

When A has the form "There is a copy of H" for a fixed graph H (where here copy is not necessarily an induced copy) then Spec (A) was found by Erdős and Rényi.

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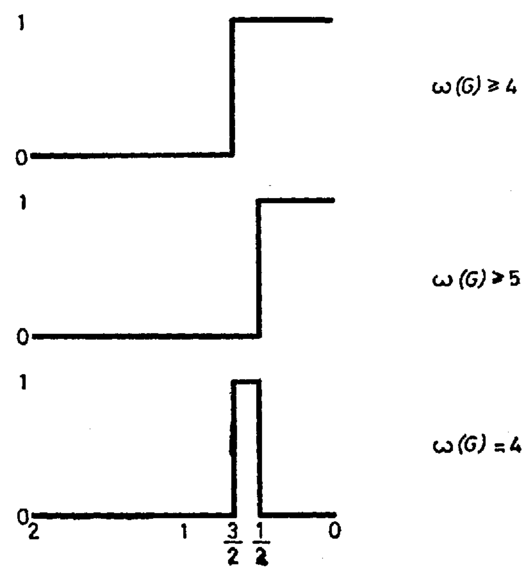

Fig. 1

In a nutshell, Spec $(A)=\{\alpha\}$ where $\alpha=v/e$ and v, e are the number of vertices and edges respectively of H *unless* H has a subgraph with a lower such ratio in which case $\alpha=v'/e'$ where H' with v' vertices, e' edges is the subgraph with the lowest such ratio. Letting, for example, A be "There is a K_4" and B be "There is a K_5", Spec $(A)=\{2/3\}$ ans Spec $(B)=\left\{\frac{1}{2}\right\}$. Indeed, for all rational α, $0<\alpha\leqq 1$ there is an A of this form with Spec $(A)=\{\alpha\}$. When A, B have disjoint spectra Spec $(A\vee B)=$Spec $(A)\cup$ $\cup$Spec (B). This is illustrated in Figure 1 where the probability of $G(n, n^{-\alpha})$ having various properties is graphed versus α and the origins of the term "spectrum" become apparent. By taking a finite sequence of exclusive ors $C=A_1\vee\ldots\vee A_n$ we may find explicit first order C having as spectra any desired *finite* set $\{\alpha_1, \ldots, \alpha_n\}$ of rational numbers in $(0, 1]$.

In [1] we showed that Spec (A) consists only of rational numbers and is scattered of finite rank. Using those methods we further may show that Spec (A) is *well-ordered* under $>$. That is, Spec (A) must be a well ordered set of rational numbers of order type less than ω^ω. (Note that as $p=n^{-\alpha}$"$>$" is the "natural" ordering from empty to full.)

From the examples above it is not clear that Spec (A) could be an infinite set. In [2] an example was given of an A with infinite spectrum. The interpretation of that A gave an arithmeticization of a fragment of the positive integers. In section 1. we give a simpler example which uses only the notion of parity. In section 2. we show that for all i there exists an A whose spectrum has order type at least ω^i. It is our hope that these methods will be extended to give a full characterization of the possible spectra of first order sentences.

1. An Infinite Spectrum

Here we give a sentence A with $\operatorname{Spec}(A)=\overline{\left\{\frac{1}{3}+\frac{1}{k}\right\}}$. Set

$$N(x, y, z) = \{u\colon\ u\sim x\wedge u\sim y\wedge u\sim z\}$$

$$H(x, y, z)\colon\ N(x, y, z) \neq \emptyset$$

$$H^w(x, y)\colon\ H(x, y, w)$$

Let $p=n^{-1/3-\varepsilon}$ with $\frac{1}{k+1}<\varepsilon<\frac{1}{k}$. The threshold for existence of the complete bipartite $K_{3,t}$ is when $n^{3+t}p^{3t}=1$ so for the given p a.s. the maximal $|N(x, y, z)|$ in $G(n, p)$ is k. We shall create an A with the interpretation that the maximal $|N(x, y, z)|$ is even. Note that for any fixed $x, y, w\in V(G)$, $\Pr[H^W(x, y)]\sim np^3\sim n^{-3\epsilon}$.

Lemma. *Let $\{x_i, y_i\}$, $1\leqq i\leqq s$ be distinct pairs of elements of $V(G)$, though possibly with overlapping vertices. Assume $s\leqq k/3$. Then the number of w with $H^W(x_i, y_i)$, $1\leqq i\leqq s$ is a.s. $n^{1-3\epsilon s+0(1)}$. Moreover, a.s. for all choices of $\{x_i, y_i\}$ the number of such w is $n^{1-3\epsilon s+0(1)}$.*

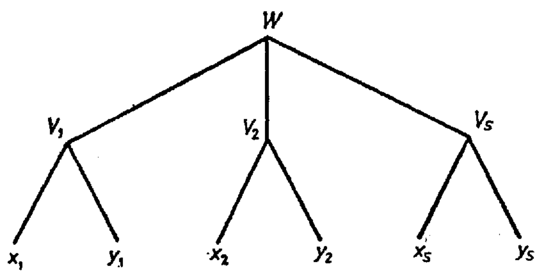

Fig. 2

Proof. In the notation of [2] the rooted graph with roots $x_1, y_1, \ldots, x_s, y_s$ and non-roots $w, v_1, \ldots, v_s$ with v_i adjacent to w, x_i, y_i is a hinged extension for $\alpha=\frac{1}{3}+\varepsilon$ and so this result follows from Theorem 3 and Lemma 4 of [2]. ▮

Theorem (Universal Property). *Let $S\subset V(G)$, $|S|=50k$ and let H be a graph on S with $s\leqq k/3$ edges. Then there are $n^{1-3\epsilon s+0(1)}$ w so that H^W on S is H.*

Proof. The lemma gives that there are $n^{1-3\epsilon s+0(1)}$ w so that H^W on S contains H. For any of the $0(1)$ graphs H' on S consisting of H and one more edge there are only $n^{1-3\epsilon(s+1)+0(1)}=o(n^{1-3\epsilon s+0(1)})$ w with H^W containing H', deleting these give the desired w. ▮

Remark. $50k$ may be replaced by any c_k.

Corollary (Extended Universal Property). *Let $S\subset V(G)$, $|S|=50k$. Let H be a graph on S with $s\leqq 10k$ edges. Then there exist $w_1, \ldots, w_{30}$ so that, on S, H is the union of the H^{W_i}.*

Proof. Partition H into $H_1 \dots H_{30}$, each with at most $k/3$ edges and for each H_i let w_i give H_i by the Theorem.

Notation. For any set $W=\{w_1, \dots, w_c\}$ we write $H^W(x,y)$ as shorthand for $H^{w_1}(x,y) \vee \dots \vee H^{w_c}(x,y)$. We write $(\exists_W |W|=c)\dots$ as shorthand for $(\exists_{w_1 w_2 \dots w_c})$. This notation can only be employed when c is a constant.

Now let S, T be sets whose membership is given by first order statements. We write

$$\text{BIGGER}\,(S,T)\colon\ (\exists_w |w| = 30)(\forall_{x\in T-S} \exists !_{y\in S-T} H^W(x,y) \wedge$$
$$\wedge (\forall_{x,x'\in T-S} \forall_{y\in S-T} H^W(x,y) \wedge H^W(x',y) \Rightarrow x = x').$$

If BIGGER (S,T) then H^W defines an injection from $T-S$ to $S-T$ and so $|T-S| \leqq |S-T|$ and thus $|S| \geqq |T|$. Crucial is the partial converse. Suppose $|T| \leqq |S| \leqq 5k$. Let $f\colon T-S \to S-T$ be an injection and let $H=\{(x, f(x))\colon\ x\in T-S\}$. Then H has at most $5k$ edges and by the Extended Universal Property there is a W so that H^W is H on $S\cup T$, hence BIGGER (S,T). To summarize: If $|S|, |T| \leqq 5k$ then BIGGER (S,T) if and only if S is bigger (or equal) to T.

Similarly we write

$$\text{EVEN}\,(S)\colon\ (\exists_w |w| = 30) \forall_{x\in S} \forall \exists !_{y\in S}\, y \neq x \wedge H^W(x,y)$$

If EVEN (S), H^W defines a matching on S so $|S|$ is even. If $|S| \leqq 5k$ and $|S|$ is even let H be any matching on S. By the Extended Universal Property there is a W with $H^W=H$ so EVEN (S). To summarize: When $S \leqq 5k$ EVEN (S) if and only if S has even size. Now write

$$\text{MAX}\,(x,y,z)\colon\ (x')(y')(z')\,\text{BIGGER}\,[N(x,y,z), N(x',y',z')]$$

In $G(n,p)$ a.s. $\max |N(x,y,z)| = k$. Thus BIGGER is bigger (or equal) and MAX (x,y,z) holds if and only if $|N(x,y,z)| = k$. Now we give our sentence.

$$A\colon\ (\exists_{x,y,z})[\text{MAX}\,(x,y,z) \wedge \text{EVEN}\,(N(x,y,z))].$$

As EVEN is even in this range A holds a.s. when k is even and $\neg A$ holds a.s. when k is odd so $\frac{1}{3}+\frac{1}{k} \in \text{Spec}\,(A)$ and, as spectra are always closed, $\frac{1}{3} \in \text{Spec}\,(A)$. We omit the argument, it not being critical, that these are precisely the points of Spec (A).

2. Spectra of Order ω^i

In this section we find by induction a sentence A_i with spectrum of order type at least ω^i. For convenience of notation we drop the index i from the predicates and sets A, EXT, UNIV, N, BIGGER, MAX, EVEN to be defined in this section. We define a critical 4-ary predicate EXT $[x,y,z,u)$. (For $i=1$ in section 1. this would be $u \sim x \wedge u \sim y \wedge u \sim z$.) For convenience we use auxilliary predicate

$$\text{UNIV}\,[x,y,z]\colon\ (\exists u)\,\text{EXT}\,[x,y,z,u]$$

and set

$$\text{N}\,[x,y,z] = \{u\colon \text{EXT}\,[x,y,z,u)\}.$$

Let $H((V(H), E(H))$ be a graph on a subset of $[n]$, the vertex set of the random graph $G(n, p)$. (H is not necessarily a subgraph of G.) We say w represents H if for all $x, y \in V(H)$

$$\{x, y\} \in E(H) \Leftrightarrow \text{UNIV}\,[x, y, w].$$

Let $3 \ll a_1 \ll a_2 \ll \ldots \ll a_i$ and let ε be sufficiently small, where "$\ll$" and "sufficiently small" may be explicitly defined. Let

$$p = n^{\alpha}, \quad \alpha = \frac{1}{3} + \frac{2}{9a_1} + \frac{2}{9a_1 a_2} + \ldots + \frac{2}{9a_1 \ldots a_i}(1+\varepsilon).$$

Our induction hypothesis (on i) is that in $G(n, p)$ a.s.

(1) All H with at most $50a_i$ vertices and at most $\frac{1}{2} a_i$ edges are represented by some w.

(2) $\max |N[x, y, z]| = 3a_i - 1$.

(3) For all t, $0 \leqq t \leqq 3a_i - 1$ there are arbitrarily many disjoint x, y, z with $|\mathrm{N}[x, y, z]| = t$.

Suppose these properties hold for i. We first give a concenient technical extension of (1). We say $W = \{w_1, \ldots, w_{20}\}$ represents $H = (V(H), E(H))$ if for all $x, y \in V(H)$, $\{x, y\} \in E(H) \Leftrightarrow \text{UNIV}\,[x, y, w_1] \vee \ldots \vee \text{UNIV}\,[x, y, w_{20}]$. Let H be any graph on at most $50a_i$ vertices and at most $10a_i$ edges. Decompose H into twenty graphs, each with at most $\frac{1}{2} a_i$ edges, and for each find, by (1), a w representing it. Then W will represent H. That is,

(1′) All H with at most $50a_i$ vertices and at most $10a_i$ edges are represented by some 20-set W.

We write $(\exists_W)$ as shorthand for $(\exists_{w_1 \ldots w_{20}})$ and UNIV^W for the graph it represents. Define

$$\text{BIGGER}\,[S, T]\colon (\exists_W)\, \text{UNIV}^W \text{ gives an injection from } T - S \text{ to } S - T.$$

Then when $|S|$, $|T| \leqq 5a_i$, $\text{BIGGER}\,[S, T] \Leftrightarrow |S| \geqq |T|$. Define

$$\text{MAX}\,[x, y, z]\colon (x')(y')(z')\ \text{BIGGER}\,[N[x, y, z], N[x', y', z']].$$

From (2), $\max |N[x, y, z)| = 3a_i - 1$ so BIGGER is bigger (or equal) on these sets and thus $\text{MAX}[x, y, z] \Leftrightarrow |N[x, y, z]| = 3a_i - 1$. Define

$$\text{EVEN}\,[S]\colon (\exists_W)\, \text{UNIV}^W \text{ is a matching on } S.$$

Then for sets of size at most $20a_i$, $\text{EVEN}\,[S] \Leftrightarrow |S|$ is even. Now define the sentence

$$A\colon (\exists_{x, y, z})\, \text{MAX}\,[x, y, z] \wedge \text{EVEN}\,[N[x, y, z]].$$

When a_i is odd $3a_i - 1$ is even and a.s. A. When a_i is even, $3a_i - 1$ is odd and a.s. $\neg A$. As this holds for all ε sufficiently small there must ben an $\alpha \in \text{Spec}\,(A)$ with

$$\frac{1}{3} + \frac{2}{9a_1} + \ldots + \frac{2}{9a_1 \ldots a_i} \geqq \alpha \geqq \frac{1}{3} + \frac{2}{9a_1} + \ldots + \frac{2}{9a_1 \ldots a_{i-1}(a_i + 1)}$$

and so $\text{Spec}\,(A)$ must have order type at least ω^i.

We now use UNIV^W to express various sizes of sets. A set S has a_i elements if there exist x, y, z so that MAX $[x, y, z]$ and MAX $[x, y, z] \cap S = \emptyset$ and there exists W so that UNIV^W on $S \times N[x, y, z]$ has degree 3 for all vertices of S except one with degree 2 and degree 1 for all vertices of $N[x, y, z]$. As we are using strong induction we may also express that S has a_j elements for any $j \leq i$. We say that S has $\frac{1}{2} a_i$ elements if there exist x, y, z so that $N[x, y, z]$ has a_i elements (described above), is disjoint from S, and there is a W so that UNIV^W on $S \times N[x, y, z]$ has degree 2 for all elements of S and degree 1 for all elements of $N[x, y, z]$. A set S has $a_t - 1$ elements if there exists $x \notin S$ so that $S \cup \{x\}$ has a_t elements. The property that a set has $a_t - c$ elements can be similarly expressed for any constant c.

We now define $\mathrm{EXT}_{i+1}[x, y, z, u]$. For notational convenience we let EXT*, UNIV*, N* denote these sets and predicates with index $i+1$. The predicate UNIV*$[x$ $y, z]$ will be an extension statement with

$$v = \frac{3}{2} a_i \dots a_1 + a_i \dots a_2 + a_i \dots a_3 + \dots + a_i + 1$$

vertices (excluding $x\ y, z$) and

$$e = \frac{9}{2} a_i \dots a_1$$

edges. We illustrate the case $i+1=4$ in Figure 3.

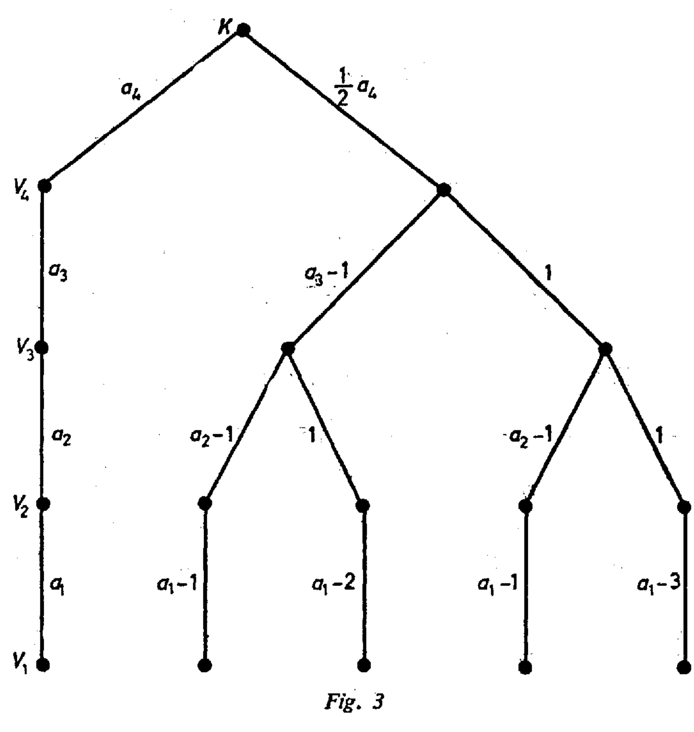

Fig. 3

The values on the edges indicate the degree of the upper level to the lower level. The bottom nodes all are joined to two of x, y, z. We read off the tree to give UNIV*. Explicitly (letting L, left, and R, right, serve as markers for where we are)
$\mathrm{UNIV}_4[x, y, z]$: There exists u so that

(L) There exist $a_4 v_4$'s which are adjacent to $a_3 v_3$'s which are adjacent to $a_2 v_2$'s which are adjacent to $a_1 v_1$'s which are adjacent to two of x, y, z

and

(R) There exist $\frac{1}{2} a_4 v_4$'s which

(RL) are adjacent to a_3-1v_3's which

(RLL) are adjacent to a_2-1v_2's which are adjacent to a_1-1v_1's which are adjacent to two of x, y, z

and

(RLR) are adjacent to one v_2 which is adjacent to a_1-2v_1's which are adjacent to two of x, y, z

and

(RR) is adjacent to one v_3 which

(RRL) is adjacent to a_2-1v_2's which are adjacent to a_1-1v_1's adjacent to two of x, y, z

and

(RRR) is adjacent to one v_2 which is adjacent to a_1-3v_1's adjacent to two of x, y, z.

We omit the general case, which follows the same pattern. The values v, e are given by a straightforward calculation.
Let $a_{i+1} \gg a_i$, let ε be sufficiently small, and set

$$\alpha = \frac{1}{3}+\frac{2}{9a_1}+\dots+\frac{2}{9a_1\dots a_i}+\frac{2(1+\varepsilon)}{9a_1\dots a_{i+1}}.$$

Note that

$$n^v p^e = n^{o-\alpha e} = n**\left[\frac{-1}{a_{i+1}}(1+\varepsilon)\right].$$

We first show (1) for UNIV* and $i+1$. Let $q \leqq a_{i+1}/2$ (actually, $q<a_{i+1}$ would do) and let $x_1, y_1, \dots, x_q, y_q$ be distinct pairs. The number of w with $\mathrm{UNIV}^*[x_s, y_s, w]$, $1 \leqq s \leqq q$ is the number of extensions of $x_1, \dots, y_q$ by an extension with $1+qv$ vertices and qe edges. The extension is hinged so the number of w is within a constant factor of the number of extensions which is asymptotically

$$n^{1+qv} p^{qe} = n**\left[1-\frac{7}{a_{i+1}}(1+\varepsilon)\right].$$

For any H on $50a_{i+1}$ vertices and q edges there are this many w with UNIV^W containing H. For each H' consisting of H and one more edge the number of w with UNIV^W containing H' has a smaller exponent and so is negligible, even when multiplied by the $0(1)$ possible H'. Thus for asymptotically this many w UNIV^W respresents H, giving (1).

The number of x, y, z with $|Nc(x, y, z)| \geqq s$ is the number of copies of a graph with $3+sv$ vertices and se edges. This graph is balanced. The number of copies is asymptotically $n^{3+sv}p^{se} = n**\left[3-\frac{s}{a_{i+1}}(1+\varepsilon)\right]$. As the exponent decreases in s for each s there will be asymptotically this many x, y, z with $|N^*(x, y, z)| = s$, giving (3). Choosing ε sufficiently small the exponent will be positive for $s = 3a_{i+1}-1$ so there will be x, y, z with $|N^*(x, y, z)| = 3a_{i+1}-1$ but for arbitrarily small ε the exponent is negative for $s = 3a_{i+1}$ and so such graphs a.s. do not exist. This shows (2), completing the induction.

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